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EXISTENCE OF INTEGRALS AND THE SOLUTION OF INTEGRAL EQUATIONS

BY

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ABSTRACT. Functions are from R to N or $R \times R$ to N, where R denotes the real numbers and N denotes a normed complete ring. If S, T and G are functions from $R \times R$ to N, each of $S(p^-, p)$, $S(p^-, p^-)$, $T(p^-, p)$ and $T(p^-, p^-)$ exists for $a < p \leq b$, each of $S(p, p^+)$, $S(p^+, p^+)$, $T(p, p^+)$ and $T(p^+, p^+)$ exists for $a \leq p < b$, G has bounded variation on $[a, b]$ and $\int_a^b G$ exists, then each of

$$\int_a^b S\left[G - \int G\right]T \quad \text{and} \quad \int_a^b S\left[1 + G - \prod(1 + G)\right]T$$

exists and is zero. These results can be used to solve integral equations without the existence of integrals of the form

$$\int_a^b \left|G - \int G\right| = 0 \quad \text{and} \quad \int_a^b \left|1 + G - \prod(1 + G)\right| = 0.$$

This is demonstrated by solving the linear integral equation

$$f(x) = h(x) + (LR)\int_a^x (fG + fH)$$

and the Riccati integral equations

$$f(x) = w(x) + (LRLR)\int_a^x (fH + Gf + fKf)$$

without the existence of the previously mentioned integrals.

Introduction. The existence of integrals of the form

$$\text{(i)} \qquad \int_a^b \left|G - \int G\right| = 0 \quad \text{and} \quad \int_a^b \left|1 + G - \prod(1 + G)\right| = 0$$

can be very useful in studying the solution of integral equations and integral inequalities. This has been demonstrated in papers by B. W. Helton [7]–[12] and the author [15]. However, it is possible to construct functions such that the integrals in (i) do not exist. This has been done by W. D. L. Appling [2, Theorem 2, p. 155] and the author [13, pp. 153–154].

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The purpose of this paper is to present results for use in solving integral equations in the event that the integrals in (i) do not exist. Two theorems of this nature are given. First, if S, T and G are functions from $R \times R$ to N, each of $S(p^-,p)$, $S(p^-,p^-)$, $T(p^-,p)$ and $T(p^-,p^-)$ exists for $a < p \leq b$, each of $S(p,p^+)$, $S(p^+,p^+)$, $T(p,p^+)$ and $T(p^+,p^+)$ exists for $a \leq p < b$, G has bounded variation on $[a, b]$ and $\int_a^b G$ exists, then $\int_a^b S[G - \int G]T$ exists and is zero. Second, if S, T and G are defined as before, then

$$\int_a^b S\Big[1 + G - \prod(1 + G)\Big]T$$

exists and is zero. In the statement of these results, the requirement that $\int_a^b G$ exist can be replaced by the requirement that ${}_x\prod^y(1 + G)$ exist for $a \leq x < y \leq b$ (see Lemma 3).

To demonstrate the use of the results stated in the preceding paragraph, the linear integral equation

$$\text{(ii)} \qquad f(x) = h(x) + (LR)\int_a^x (fG + fH)$$

and the Riccati integral equation

$$\text{(iii)} \qquad f(x) = w(x) + (LRLR)\int_a^x (fH + Gf + fKf)$$

are solved without requiring the existence of the integrals in (i). Further, the solutions are obtained with previous restrictions on h and w relaxed from bounded variation to quasi-continuity.

The solution presented for the linear integral equation in (ii) extends results by B. W. Helton [7, Theorem 5.1, p. 310] and the author [15, Theorem], [16, Theorem 1] since the integrals in (i) are not required to exist and h is not required to have bounded variation. The ideas behind the existence results in the second paragraph are contained in a previous paper by the author [15], where the integral equation in (ii) is solved without requiring the existence of the integrals in (i). However, at the time, the nature of these results and their use in solving integral equations were not recognized. The presentation here is more direct than that in [15] and also more general since h is not required to have bounded variation.

The solution presented for the Riccati integral equation in (iii) extends a result by B. W. Helton [11, Theorems 3.2, 3.3]. The result given here is more general since the integrals in (i) are not required to exist and w is not required to have bounded variation.

This paper is part of a sequence of results initiated by two papers by H. S. Wall [42]–[43]. There, in response to a paper by P. Puig Adam [1], Wall developed techniques involving continuous continued fractions and harmonic matrices for the solution of integral equations. In turn, these results have been extended in various directions by J. S. Mac Nerney [29]–[34], J. W. Neuberger

[36]–[38], T. H. Hildebrandt [22], J. R. Dorroh [5], B. W. Helton [7]–[12], D. B. Hinton [23], J. V. Herod [18]–[21], C. W. Bitzer [3]–[4], D. L. Lovelady [24]–[28], J. A. Reneke [39]–[41], G. F. Webb [44]–[46], and the author [14]–[17]. For early background on product integration, the reader is referred to P. R. Masani [35] and F. R. Gantmacher [6, Chapter 24].

Definitions. All integrals and definitions are of the subdivision-refinement type, and functions are from R to N or $R \times R$ to N, where R denotes the set of real numbers and N denotes a ring which has a multiplicative identity element represented by 1 and a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$. At times, we consider 2×2 and 1×2 matrices composed of elements of N. Then, we use $\max_{i=1,2}\Sigma_{j=1}^{2}|a_{ij}|$ and $\Sigma_{i=1}^{2}|a_i|$ as our norms and denote them by $\|\cdot\|$. Lower case letters are used to denote functions from R to N, and capital letters are used to denote functions from $R \times R$ to N. Boldface letters are used to represent matrix functions. Functions on $R \times R$ are assumed to be defined only for elements $\{a, b\}$ of $R \times R$ such that $a < b$. If $\{x_i\}_{i=0}^{n}$ is a subdivision of $[a, b]$, then

$$f_i = f(x_i) \quad \text{and} \quad G_i = G(x_{i-1}, x_i).$$

Further if $\{x_{ij}\}_{j=0}^{n(i)}$ is a subdivision of $[x_{i-1}, x_i]$, then $G_{ij} = G(x_{i,j-1}, x_{ij})$ and $f_{ij} = f(x_{ij})$.

The statement that $\int_a^b G$ exists means there exists an element L of N such that, if $\varepsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^{n}$ is a refinement of D, then

$$\left|L - \sum_{i=1}^{n} G_i\right| < \varepsilon.$$

The statement that ${}_a\Pi^b(1 + G)$ exists means there exists an element L of N such that, if $\varepsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^{n}$ is a refinement of D, then

$$\left|L - \prod_{i=1}^{n} (1 + G_i)\right| < \varepsilon.$$

The statement that $(LR)\int_a^b(fG + fH)$ exists means $\int_a^b C$ exists, where $C(x,y) = f(x)G(x,y) + f(y)H(x,y)$ for $a \leqslant x < y \leqslant b$. The integral $(LRLR)\int_a^b(fH + Gf + fKf)$ is defined similarly.

The statements that G is bounded on $[a, b]$, G has bounded variation on $[a, b]$ and G is in $OP°$ on $[a, b]$ mean there exist a subdivision D of $[a, b]$ and a number B such that, if $\{x_i\}_{i=0}^{n}$ is a refinement of D, then

(1) $|G_i| < B$ for $i = 1, 2, \ldots, n$,

(2) $\Sigma_{i=1}^{n}|G_i| < B$, and

(3) $|\Pi_{i=r+1}^{s}(1 + G_i)| < B$ for $0 \leqslant r < s \leqslant n$,

respectively.

The statement that G is in OM° on $[a, b]$ means ${}_x\Pi^y(1 + G)$ exists for $a \leqslant x < y \leqslant b$ and $\int_a^b |1 + G - \Pi(1 + G)|$ exists and is zero. The statement that G is in OM^* on $[a, b]$ means ${}_x\Pi^y(1 + G)$ exists for $a \leqslant x < y \leqslant b$ and, if $\varepsilon > 0$, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D and $0 \leqslant p < q \leqslant n$, then

$$\left| {}_{x_p}\Pi^{x_q}(1 + G) - \prod_{i=p+1}^{q} (1 + G_i) \right| < \varepsilon.$$

The symbols $G(p^-, p)$, $G(p^-, p^-)$, $G(p, p^+)$ and $G(p^+, p^+)$ are used to denote

$$\lim_{x \to p^-} G(x, p), \quad \lim_{x, y \to p^-} G(x, y), \quad \lim_{x \to p^+} G(p, x)$$

and

$$\lim_{x, y \to p^+} G(x, y),$$

respectively. Further, the function h is quasi-continuous on $[a, b]$ only if $\lim_{x \to p^-} h(x)$ exists for $a < p \leqslant b$ and $\lim_{x \to p^+} h(x)$ exists for $a \leqslant p < b$. For convenience in notation, let $G \in S_1$ on $[a, b]$ only if $G(p^-, p^-)$ exists and is zero for $a < p \leqslant b$ and $G(p^+, p^+)$ exists and is zero for $a \leqslant p < b$, and let $G \in S_2$ on $[a, b]$ only if $G(p^-, p)$ exists for $a < p \leqslant b$ and $G(p, p^+)$ exists for $a \leqslant p < b$.

Results. The two existence theorems are presented first. One lemma is needed.

LEMMA 1. *If S, T and G are functions from $R \times R$ to N, each of $S(p^-, p)$, $S(p^-, p^-)$, $T(p^-, p)$ and $T(p^-, p^-)$ exists for $a < p \leqslant b$, each of $S(p, p^+)$, $S(p^+, p^+)$, $T(p, p^+)$ and $T(p^+, p^+)$ exists for $a \leqslant p < b$, G has bounded variation on $[a, b]$ and $\int_x^y G$ exists and is zero for $a \leqslant x < y \leqslant b$, then $\int_a^b SGT$ exists and is zero.*

PROOF. Let $\varepsilon > 0$. It follows from the existence of the limits involving S and T by an application of the covering theorem that S and T are bounded on $[a, b]$. Thus, since G has bounded variation on $[a, b]$, there exist a subdivision D_1 of $[a, b]$ and a number B such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then

$$\sum_{i=1}^{n} |G_i|\,|T_i| < B \quad \text{and} \quad \sum_{i=1}^{n} (|S_i| + 1)|G_i| < B.$$

Again, it follows from the existence of the limits involving S and T by an application of the covering theorem that there exist a subdivision $\{r_j\}_{j=0}^p$ of $[a, b]$ and sequences $\{s_j\}_{j=1}^p$ and $\{t_j\}_{j=1}^p$ of elements of N such that, if $1 \leqslant j \leqslant p$ and $r_{j-1} < x < y < r_j$, then

$$|S(x, y) - s_j| < \varepsilon(4B)^{-1}, \qquad |s_j| < |S(x, y)| + 1$$

and

$$|T(x,y) - t_j| < \varepsilon(4B)^{-1}.$$

Since G has bounded variation on $[a, b]$, S and T are bounded on $[a, b]$, $G(p,p^+)$ exists and is zero for $a \leq p < b$ and $G(p^-,p)$ exists and is zero for $a < p \leq b$, it follows that there exist subdivisions $\{u_j\}_{j=0}^{p+1}$ and $\{v_j\}_{j=0}^{p+1}$ of $[a, b]$ such that $r_{j-1} < u_j < v_j < r_j$ for $j = 1, 2, \ldots, p$ and also such that

$$\left|\sum_{j=1}^{p+1}\sum_{k=1}^{n(j)} S_{jk}G_{jk}T_{jk}\right| < \varepsilon/4$$

for refinements $\{x_{jk}\}_{k=0}^{n(j)}$ of $\{v_{j-1}, r_{j-1}, u_j\}$.

For $j = 1, 2, \ldots, p$, there exists a subdivision E_j of $[u_j, v_j]$ such that, if $\{x_i\}_{i=0}^{n}$ is a refinement of E_j, then

$$\left|\sum_{i=1}^{n} s_j G_i t_j\right| < \varepsilon(4p)^{-1}.$$

This is true since $\int_{u_j}^{v_j} s_j G t_j$ exists and is zero.

Let D denote the subdivision

$$D_1 \cup D_2 \bigcup_{j=1}^{p} E_j \cup \{r_j\}_{j=0}^{p}$$

of $[a, b]$. Suppose $\{x_i\}_{i=0}^{n}$ is a refinement of D. For $j = 1, 2, \ldots, p$, let P_j denote the set such that $i \in P_j$ only if $u_j < x_i < v_j$. Further, let Q denote $\{i\}_{i=1}^{n} - \cup_{j=1}^{p} P_j$. Now,

$$\begin{aligned}\left|\sum_{i=1}^{n} S_iG_iT_i\right| &\leq \sum_{j=1}^{p}\left|\sum_{i\in P_j} S_iG_iT_i\right| + \left|\sum_{i\in Q} S_iG_iT_i\right| \\ &< \sum_{j=1}^{p}\left|\sum_{i\in P_j}(S_i - s_j + s_j)G_i(T_i - t_j + t_j)\right| + \varepsilon/4 \\ &\leq \sum_{j=1}^{p}\left|\sum_{i\in P_j} s_jG_it_j\right| + \sum_{j=1}^{p}\sum_{i\in P_j}|s_j|\,|G_i|\,|T_i - t_j| \\ &\quad + \sum_{j=1}^{p}\sum_{i\in P_j}|S_i - s_j|\,|G_i|\,|T_i| + \varepsilon/4 \\ &< p\left[\varepsilon(4p)^{-1}\right] + \left[\varepsilon(4B)^{-1}\right]B + \left[\varepsilon(4B)^{-1}\right]B + \varepsilon/4 \\ &= \varepsilon.\end{aligned}$$

Therefore, $\int_a^b SGT$ exists and is zero. This completes the proof of Lemma 1.

THEOREM 1. *If S, T and G are functions from $R \times R$ to N, each of $S(p^-,p)$, $S(p^-,p^-)$, $T(p^-,p)$ and $T(p^-,p^-)$ exists for $a < p \leq b$, each of $S(p,p^+)$,*

$S(p^+, p^+)$, $T(p, p^+)$ and $T(p^+, p^+)$ exists for $a \leqslant p < b$, G has bounded variation on $[a, b]$ and $\int_a^b G$ exists, then

$$\int_a^b S\left[G - \int G\right] T$$

exists and is zero.

PROOF. This theorem follows as a corollary to Lemma 1.

THEOREM 2. *If S, T and G are functions from $R \times R$ to N, each of $S(p^-, p)$, $s(p^-, p^-)$, $T(p^-, p)$ and $T(p^-, p^-)$ exists for $a < p \leqslant b$, each of $S(p, p^+)$, $S(p^+, p^+)$, $T(p, p^+)$ and $T(p^+, p^+)$ exists for $a \leqslant p < b$, G has bounded variation on $[a, b]$ and ${}_x\Pi^y(1 + G)$ exists for $a \leqslant x < y \leqslant b$, then*

$$\int_a^b S\left[1 + G - \prod(1 + G)\right] T$$

exists and is zero.

PROOF. This theorem follows as a corollary to Lemma 1.

The solution of the linear integral equation is now presented. Several additional lemmas are needed.

LEMMA 2. *If S, T and G are functions from $R \times R$ to N, each of $S(p^-, p)$, $S(p^-, p^-)$, $T(p^-, p)$ and $T(p^-, p^-)$ exists for $a < p \leqslant b$, each of $S(p, p^+)$, $S(p^+, p^+)$, $T(p, p^+)$ and $T(p^+, p^+)$ exists for $a \leqslant p < b$, G has bounded variation on $[a, b]$ and $\int_a^b G$ exists, then $\int_a^b SGT$ exists* [14, *Theorem* 5].

LEMMA 3. *If G is a function from $R \times R$ to N and G has bounded variation on $[a, b]$, then $\int_a^b G$ exists if and only if ${}_x\Pi^y(1 + G)$ exists for $a \leqslant x < y \leqslant b$* [14, *Theorem* 4].

LEMMA 4. *If h is a function from R to N, H and L are functions from $R \times R$ to N, h is quasi-continuous on $[a, b]$, H and $L - 1$ have bounded variation on $[a, b]$, $\int_a^b H$ exists and $\int_a^b (L - 1)$ exists, then*

$$(R)\int_a^b dhL\left[{}_v\prod^b(1 + H)\right]$$

exists.

PROOF. For convenience in notation, let

$$S_1(x, y) = (R)\int_x^y dh(L - 1)\left[{}_v\prod^y(1 + H)\right]$$

$$S_2(x, y) = \lim_{t \to x^+} (LR)\int_t^y h(u)H\left[{}_v\prod^y(1 + H)\right]$$

and

$$S_3(x, y) = \lim_{t \to x^+} h(x)\left[{}_t\prod^y(1 + H)\right]$$

for $a < x < y < b$. The existence of $S_1(x, y)$ and $S_2(x, y)$ can be established by using Lemma 2. For simplicity, S_1, S_2 and S_3 are used to denote $S_1(a, b)$, $S_2(a, b)$ and $S_3(a, b)$, respectively. In the following, it is shown that the desired integral exists and is

$$S_1 + S_2 + h(b) - S_3.$$

Let $\varepsilon > 0$.

If $\{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$, then

$$\begin{aligned}\sum_{i=1}^n dh_i L_i\left[{}_{x_i}\prod^b (1 + H)\right] &= \sum_{i=1}^n dh_i[L_i - 1]\left[{}_{x_i}\prod^b (1 + H)\right] \\ &\quad + \sum_{i=2}^n h_{i-1}H_i\left[{}_{x_i}\prod^b (1 + H)\right] + h(b) \\ &\quad - h(a)\left[{}_{x_1}\prod^b (1 + H)\right] + \sum_{i=1}^{n-1} h_i e_i\left[{}_{x_{i+1}}\prod^b (1 + H)\right],\end{aligned}$$

where

$$e_i = {}_{x_i}\prod^{x_{i+1}} (1 + H) - (1 + H_{i+1})$$

for $i = 1, 2, \ldots, n - 1$. Now, by employing the preceding relation, we have that

$$\left|\left\{\sum_{i=1}^n dh_i L_i\left[{}_{x_i}\prod^b (1 + H)\right]\right\} - \{S_1 + S_2 + h(b) - S_3\}\right|$$

$$\text{(a)} \qquad < \left|\sum_{i=1}^n dh_i[L_i - 1]\left[{}_{x_i}\prod^b (1 + H)\right] - S_1\right|$$

$$\text{(b)} \qquad + \left|\sum_{i=2}^n h_{i-1}H_i\left[{}_{x_i}\prod^b (1 + H)\right] - S_2\right|$$

$$\text{(c)} \qquad + \left|S_3 - h(a)\left[{}_{x_1}\prod^b (1 + H)\right]\right|$$

$$\text{(d)} \qquad + \left|\sum_{i=1}^{n-1} h_i e_i\left[{}_{x_{i+1}}\prod^b (1 + H)\right]\right|.$$

It follows from the existence of S_1, S_2 and S_3 that there exists a subdivision D_1 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then each of the expressions in (a), (b) and (c) is less than $\varepsilon/4$. Further, it follows from Theorem 2 that there exists a subdivision D_2 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then the expression in (d) is less than $\varepsilon/4$. Let D denote the subdivision $D_1 \cup D_2$ of $[a, b]$ and suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Then,

$$\left|\left\{\sum_{i=1}^n dh_i L_i\left[{}_{x_i}\prod^b (1 + H)\right]\right\} - \{S_1 + S_2 + h(b) - S_3\}\right|$$
$$< \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon.$$

Therefore, the desired integral exists and is

$$S_1 + S_2 + h(b) - S_3.$$

This completes the proof of Lemma 4.

LEMMA 5. *If G is a function from $R \times R$ to N, G has bounded variation on $[a, b]$ and ${}_x\prod^y(1 + G)$ exists for $a \leqslant x < y \leqslant b$, then G is in OM^* on $[a, b]$ [14, Theorem 1].*

LEMMA 6. *If $\varepsilon > 0$, h is a function from R to N, H and L are functions from $R \times R$ to N, h is quasi-continuous on $[a, b]$, H and $L - 1$ have bounded variation on $[a, b]$, $\int_a^b H$ exists and $\int_a^b(L - 1)$ exists, then there exists a subdivision D of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then*

$$\left|\sum_{i=1}^n dh_i L_i\left[\prod_{j=i+1}^n (1 + H_j)\right] - (R)\int_a^b dhL\left[{}_v\prod^b (1 + H)\right]\right| < \varepsilon.$$

PROOF. It is established in Lemma 4 that

$$(R)\int_a^b dhL\left[{}_v\prod^b (1 + H)\right] = S_1 + S_2 + h(b) - S_3,$$

where S_1, S_2 and S_3 are defined in Lemma 4. Also, if $\{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$, then

$$\begin{aligned}\sum_{i=1}^n dh_i L_i&\left[\prod_{j=i+1}^n (1 + H_j)\right] \\ &= \sum_{i=1}^n dh_i[L_i - 1]\left[\prod_{j=i+1}^n (1 + H_j)\right] \\ &\quad + \sum_{i=2}^n h_{i-1}H_i\left[\prod_{j=i+1}^n (1 + H_j)\right] + h(b) \\ &\quad - h(a)\left[\prod_{j=2}^n (1 + H_j)\right].\end{aligned}$$

Now, by employing the two preceding relations, we have that

$$\left|\sum_{i=1}^{n} dh_i L_i\left[\prod_{j=i+1}^{n}(1+H_j)\right] - (R)\int_a^b dhL\left[{}_v\prod^b(1+H)\right]\right|$$

$$\text{(a)} \qquad \leqslant \left|\sum_{i=1}^{n} dh_i[L_i - 1]\left[\prod_{j=i+1}^{n}(1+H_j)\right] - S_1\right|$$

$$\text{(b)} \qquad + \left|\sum_{i=2}^{n} h_{i-1}H_i\left[\prod_{j=i+1}^{n}(1+H_j)\right] - S_2\right|$$

$$\text{(c)} \qquad + \left|S_3 - h(a)\left[\prod_{j=2}^{n}(1+H_j)\right]\right|.$$

Let $\varepsilon > 0$. Since $L - 1$ has bounded variation on $[a, b]$ and H is in OM^* on $[a, b]$ by Lemma 5, there exists a subdivision D_1 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then the expression in (a) is less than $\varepsilon/3$. Similarly, since H has bounded variation on $[a, b]$, there exists a subdivision D_2 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then the expression in (b) is less than $\varepsilon/3$. Finally, from the existence of S_3, it follows that there exists a subdivision D_3 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_3, then the expression in (c) is less than $\varepsilon/3$.

Let D denote the subdivision $D_1 \cup D_2 \cup D_3$ of $[a, b]$ and suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Then,

$$\left|\sum_{i=1}^{n} dh_i L_i\left[\prod_{j=i+1}^{n}(1+H_j)\right] - (R)\int_a^b dhL\left[{}_v\prod^b(1+H)\right]\right|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Therefore, the desired inequality is established. This completes the proof of Lemma 6.

LEMMA 7. *If h is a function from R to N, H and L are functions from $R \times R$ to N, h is quasi-continuous on $[a, b]$, H and $L - 1$ have bounded variation on $[a, b]$, $\int_a^b H$ exists and $\int_a^b (L - 1)$ exists, then $\int_a^b |dhL - I|$ exists and is zero, where*

$$I(x,y) = (R)\int_x^y dhL\left[{}_v\prod^y(1+H)\right]$$

for $a \leqslant x < y \leqslant b$.

PROOF. It is established in Lemma 4 that

$$I(x,y) = S_1(x,y) + S_2(x,y) + h(y) - S_3(x,y)$$

for $a \leqslant x < y \leqslant b$, where $S_1(x, y)$, $S_2(x, y)$ and $S_3(x, y)$ are defined in Lemma 4. Thus, if $\{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$, then

$$\sum_{i=1}^{n} |dh_i L_i - I(x_{i-1}, x_i)|$$

(a) $$< \sum_{i=1}^{n} |dh_i(L_i - 1) - S_1(x_{i-1}, x_i)|$$

(b) $$+ \sum_{i=1}^{n} |S_3(x_{i-1}, x_i) - h_{i-1} - S_2(x_{i-1}, x_i)|.$$

We now establish that the desired integral exists and is zero. Let $\varepsilon > 0$.

For any specified positive number α, there exists a subdivision $\{x_i\}_{i=0}^n$ of $[a, b]$ such that, if $x_{i-1} < u_i < v_i < x_i$ and $\{x_{ij}\}_{j=0}^{n(i)}$ is a subdivision of $[u_i, v_i]$ for $i = 1, 2, \ldots, n$, then

$$\sum_{i=1}^{n} \sum_{j=1}^{n(i)} |dh_{ij}(L_{ij} - 1)| < \alpha.$$

By using this as a starting point, it is possible to construct a subdivision D_1 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then the expression in (a) is less than $\varepsilon/2$.

We next note two things. First,

$${}_x\prod^y (1 + H) = 1 + (R)\int_x^y H\left[{}_v\prod^y (1 + H)\right]$$

for $a \leq x < y \leq b$. Second, if α is a positive number, then there exists a subdivision $\{x_i\}_{i=0}^n$ of $[a, b]$ such that, if $x_{i-1} < u_i < v_i < x_i$ for $i = 1, 2, \ldots, n$, then

$$\sum_{i=1}^{n} \left|(LR)\int_{u_i}^{v_i} [h(u_i) - h(u)] H\left[{}_v\prod^{v_i} (1 + H)\right]\right| < \alpha.$$

By employing the two preceding relations, it is possible to construct a subdivision D_2 of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then the expression in (b) is less than $\varepsilon/2$.

Let D denote the subdivision $D_1 \cup D_2$ of $[a, b]$ and suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Then,

$$\sum_{i=1}^{n} |dh_i L_i - I(x_{i-1}, x_i)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore, the desired integral exists and is zero. This completes the proof of Lemma 7.

THEOREM 3. *If f and h are functions from R to N, G, H, A and B are functions from $R \times R$ to N, $(1 - H)^{-1}$ exists and is bounded on $[a, b]$, $B = (1 - H)^{-1}$ and $A = (1 + G)(1 - H)^{-1}$ on $[a, b]$, h is quasi-continuous on $[a, b]$, G and H have bounded variation on $[a, b]$, $\int_a^b G$ exists and $\int_a^b H$ exists, then the following statements are equivalent:*

(1) *f is bounded on $[a, b]$ and $(LR)\int_a^x(fG + fH)$ exists and is $f(x) - h(x)$ for $a \leqslant x \leqslant b$, and*

(2) *$(R)\int_a^x(dhB_v\prod^x A)$ exists and is $f(x) - h(a)_a\prod^x A$ for $a \leqslant x \leqslant b$.*

Proof [(1) → (2)]. It follows that ${}_x\prod^y A$ exists for $a \leqslant x < y \leqslant b$ by employing Lemmas 2 and 3. Further, the existence of $(R)\int_a^x(dhB_v\prod^x A)$ for $a \leqslant x \leqslant b$ follows from Lemma 4. We now establish that

$$f(x) = h(a)_a\prod^x A + (R)\int_a^x\left(dhB_v\prod^x A\right)$$

for $a \leqslant x \leqslant b$. If $a = x$, the equality follows immediately.

Suppose $a < x \leqslant b$ and $\varepsilon > 0$. If D is a subdivision of $[a, x]$ and $\{x_i\}_{i=0}^n$ is a refinement of D, then by employing an iteration technique used by B. W. Helton [7, Theorem 5.1, p. 311], we have that

$$f_n = f_0\prod_{i=1}^n A_i + \sum_{i=1}^n dh_iB_i\prod_{j=i+1}^n A_j + \sum_{i=1}^n c_iB_i\prod_{j=i+1}^n A_j,$$

where

$$c_i = (LR)\int_{x_{i-1}}^{x_i}(fG + fH) - (f_{i-1}G_i + f_iH_i)$$

for $i = 1, 2, \ldots, n$. Thus,

$$\left|f(x) - \left[h(a)_a\prod^x A + (R)\int_a^x\left(dhB_v\prod^x A\right)\right]\right|$$

$$\text{(a)}\qquad \leqslant\left|h(a)\prod_{i=1}^n A_i - h(a)_a\prod^x A\right|$$

$$\text{(b)}\qquad +\left|\sum_{i=1}^n dh_iB_i\prod_{j=i+1}^n A_j - (R)\int_a^x\left(dhB_v\prod^x A\right)\right|$$

$$\text{(c)}\qquad +\left|\sum_{i=1}^n c_iB_i\prod_{j=i+1}^n A_j\right|.$$

Since ${}_a\prod^x A$ exists, there exists a subdivision D_1 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then the expression in (a) is less than $\varepsilon/3$. Further, it follows from Lemma 6 that there exists a subdivision D_2 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then the expression in (b) is less than $\varepsilon/3$. It follows from Lemma 5 that $A - 1$ is in OM^* on $[a, x]$. Now, from this and Theorem 1, it follows that there exists a subdivision D_3 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_3, then the expression in (c) is less than $\varepsilon/3$.

Let D denote the subdivision $\cup_{i=1}^3 D_i$ of $[a, x]$. Thus, if $\{x_i\}_{i=0}^n$ is a refinement of D, then

$$\left|f(x) - \left[h(a)_a\prod^x A + (R)\int_a^x \left(dhB_v\prod^x A\right)\right]\right|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Therefore, (1) implies (2).

PROOF [(2) → (1)]. It follows from the definition of f in (2) that f is bounded on $[a, b]$. The existence of $(LR)\int_a^x(fG + fH)$ for $a \leqslant x \leqslant b$ follows from Lemma 2. We now establish that

$$f(x) = h(x) + (LR)\int_a^x (fG + fH)$$

for $a \leqslant x \leqslant b$. If $a = x$, the equality follows immediately.

Suppose $a < x \leqslant b$ and $\varepsilon > 0$. If D is a subdivision of $[a, x]$ and $\{x_i\}_{i=0}^n$ is a refinement of D, then by employing an iteration technique used by B. W. Helton [7, Theorem 5.1, p. 312], we have that

$$f_n = h_n + \sum_{i=1}^n (f_{i-1}G_i + f_iH_i) + \sum_{i=1}^n f_{i-1}d_i(1 - H_i) + \sum_{i=1}^n e_i(1 - H_i),$$

where

$$d_i = {}_{x_{i-1}}\prod^{x_i} A - A_i$$

and

$$e_i = (R)\int_{x_{i-1}}^{x_i} \left(dhB_v\prod^{x_i} A\right) - dh_iB_i$$

for $i = 1, 2, \ldots, n$. Thus,

$$\left|f(x) - \left[h(x) + (LR)\int_a^x (fG + fH)\right]\right|$$

(a) $$\leqslant \left|\sum_{i=1}^n (f_{i-1}G_i + f_iH_i) - (LR)\int_a^x (fG + fH)\right|$$

(b) $$+ \left|\sum_{i=1}^n f_{i-1}d_i(1 - H_i)\right|$$

(c) $$+ \sum_{i=1}^n |e_i(1 - H_i)|.$$

Since $(LR)\int_a^x(fG + fH)$ exists, there exists a subdivision D_1 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then the expression in (a) is less than $\varepsilon/3$. Further, it follows from Theorem 2 that there exists a subdivision D_2 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then the expression in (b) is less than $\varepsilon/3$. Finally, it follows from Lemma 7 that there exists a subdivision D_3 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_3, then the expression in (c) is less than $\varepsilon/3$.

Let D denote the subdivision $\cup_{i=1}^{3} D_i$ of $[a, x]$. Thus, if $\{x_i\}_{i=0}^{n}$ is a refinement of D, then

$$\left| f(x) - \left[h(x) + (LR)\int_a^x (fG + fH) \right] \right| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Therefore, (2) implies (1). This completes the proof of Theorem 3.

The solution of the Riccati integral equation is now presented. Several additional lemmas are needed.

LEMMA 8. *If F and G are functions from $R \times R$ to N, F is in $OP°$, OM^* and $S_1 \cap S_2$ on $[a, b]$, G has bounded variation on $[a, b]$ and G is in OM^* on $[a, b]$, then $F + G$ is in $OP°$ and OM^* on $[a, b]$* [17, *Lemma* 5, *Theorem* 1].

LEMMA 9. *If F and G are functions from $R \times R$ to N, F is in $OP°$, OM^* and $S_1 \cap S_2$ on $[a, b]$, G has bounded variation on $[a, b]$ and G is in OM^* on $[a, b]$, then*

$$(LR)\int_a^b \left[{}_a\prod^r (1 + F) \right] G(r, s) \left[{}_s\prod^b (1 + F + G) \right]$$

exists and is

$${}_a\prod^b (1 + F + G) - {}_a\prod^b (1 + F).$$

PROOF. It follows from Lemma 8 that $F + G$ is in $OP°$ and OM^* on $[a, b]$. Further, it can be established by induction that, if $\{x_i\}_{i=0}^{n}$ is a subdivision of $[a, b]$, then

$$\sum_{i=1}^{n} \left[\prod_{j=1}^{i-1} (1 + F_j) \right] G_i \left[\prod_{j=i+1}^{n} (1 + F_j + G_j) \right]$$
$$= \prod_{i=1}^{n} (1 + F_i + G_i) - \prod_{i=1}^{n} (1 + F_i).$$

The existence of the desired integral can be established by using the preceding identity and the facts that G has bounded variation on $[a, b]$ and that F and $F + G$ are in $OP°$ and OM^* on $[a, b]$. This completes the proof of Lemma 9.

LEMMA 10. *If G is a function from $R \times R$ to N, G has bounded variation on $[a, b]$, ${}_x\prod^y (1 + G)$ exists for $a \leq x < y \leq b$ and $(1 + G)^{-1}$ exists and is bounded on $[a, b]$, then $[{}_x\prod^y (1 + G)]^{-1}$ exists and is bounded on $[a, b]$.*

PROOF. If A is an element of N and $|A| < 1$, then

$$(1 + A)^{-1} = 1 + \sum_{j=1}^{\infty} (-1)^j A^j.$$

By employing this relation, we can establish that, if G satisfies the conditions in the hypothesis on some interval $[u, v]$ plus the additional condition that $|G| < \frac{1}{2}$ on that interval, then $[{}_u\Pi^v(1 + G)]^{-1}$ exists and is ${}_v\Pi^u(1 + H)$, where

$$H(s, r) = \sum_{j=1}^{\infty} (-1)^j G^j(r, s)$$

for $u \leqslant r < s \leqslant v$.

Since G has bounded variation on $[a, b]$, there exists a subdivision $\{x_i\}_{i=0}^n$ of $[a, b]$ such that, if $1 \leqslant i \leqslant n$ and $x_{i-1} < u < v < x_i$, then $|G| < \frac{1}{2}$ on $[u, v]$. Hence, the existence of $[{}_x\Pi^y(1 + G)]^{-1}$ for $a \leqslant x < y \leqslant b$ can be established from the existence of $(1 + G)^{-1}$ and the result given in the previous paragraph. Further, since G has bounded variation, it also follows that this inverse is bounded on $[a, b]$. This completes the proof of Lemma 10.

LEMMA 11. *If F, G and L are functions from $R \times R$ to N, F is in $OP°$, $OM°$ and $S_1 \cap S_2$ on $[a, b]$, G has bounded variation on $[a, b]$, $\int_a^b G$ exists and*

$$L(r, s) = {}_r\Pi^s(1 + F + G) - 1 - F(r, s) - G(r, s)$$

for $a \leqslant r < s \leqslant b$, then L has bounded variation on $[a, b]$ and $\int_x^y L$ exists and is zero for $a \leqslant x < y \leqslant b$.

PROOF. It is initially established that $\int_x^y L$ exists and is zero for $a \leqslant x < y \leqslant b$. Suppose $a \leqslant x < y \leqslant b$ and $\varepsilon > 0$. By applying Lemma 9, we have that, if $\{x_i\}_{i=0}^n$ is a subdivision of $[x, y]$, then

$$\left|\sum_{i=1}^n L_i\right| = \left|\sum_{i=1}^n {}_{x_{i-1}}\Pi^{x_i}(1 + F + G) - (1 + F_i + G_i)\right|$$

$$\text{(a)} \qquad \leqslant \left|\sum_{i=1}^n {}_{x_{i-1}}\Pi^{x_i}(1 + F) - (1 + F_i)\right|$$

$$\text{(b)} \qquad + \left|\sum_{i=1}^n (LR)\int_{x_{i-1}}^{x_i} \left[{}_{x_{i-1}}\Pi^r(1 + F)\right] G(r, s)\left[{}_s\Pi^{x_i}(1 + F + G)\right] - G_i\right|.$$

Since F is in $OM°$ on $[x, y]$, there exists a subdivision D_1 of $[x, y]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then the expression in (a) is less than $\varepsilon/2$.

For any specified positive number α, there exists a subdivision $\{x_i\}_{i=0}^n$ of $[x, y]$ such that, if $1 \leqslant i \leqslant n$ and $x_{i-1} < u < v < x_i$, then

$$\left|1 - {}_u\Pi^v(1 + F)\right| < \alpha \quad \text{and} \quad \left|1 - {}_u\Pi^v(1 + F + G)\right| < \alpha.$$

Hence, it is possible to construct a subdivision D_2 of $[x, y]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then the expression in (b) is less than $\varepsilon/2$.

Let D denote the subdivision $D_1 \cup D_2$ of $[x, y]$ and suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Then,

$$\left|\sum_{i=1}^{n} L_i\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, $\int_x^y L$ exists and is zero.

It is now established that L has bounded variation on $[a, b]$. Again by applying Lemma 9, we have that, if $\{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$, then

$$\sum_{i=1}^{n} |L_i| = \sum_{i=1}^{n} \left| {}_{x_{i-1}}\prod^{x_i}(1 + F + G) - (1 + F_i + G_i)\right|$$

$$< \sum_{i=1}^{n} \left| {}_{x_{i-1}}\prod^{x_i}(1 + F) - (1 + F_i)\right|$$

(a) $$+ \sum_{i=1}^{n} \left|(LR)\int_{x_{i-1}}^{x_i} \left[{}_{x_{i-1}}\prod^{r}(1 + F)\right] G(r, s)\left[{}_{s}\prod^{x_i}(1 + F + G)\right]\right|$$

(b) $$+ \sum_{i=1}^{n} |-G_i|.$$

Since F is in $OM°$ on $[a, b]$, there exist a subdivision D_1 of $[a, b]$ and a number B_1 such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then the expression in (a) is less than B_1. Since G has bounded variation on $[a, b]$ and each of F and $F + G$ is in $OP°$ on $[a, b]$, there exist a subdivision D_2 of $[a, b]$ and numbers B_2 and B_3 such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then the expressions in (b) and (c) are less than B_2 and B_3, respectively.

Let D denote the subdivision $D_1 \cup D_2$ of $[a, b]$ and suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Then,

$$\sum_{i=1}^{n} |L_i| < B_1 + B_2 + B_3.$$

Hence, L has bounded variation on $[a, b]$. This completes the proof of Lemma 11.

THEOREM 4. *If w is a function from R to N, each of G, H and K is a function from $R \times R$ to N, w is quasi-continuous on $[a, b]$, each of G, H and K has bounded variation on $[a, b]$, $K(p, p^+)$ exists and is zero for $a \leqslant p < b$, $K(p^-, p)$ exists and is zero for $a < p \leqslant b$, $(1 - G)^{-1}$ exists and is bounded on $[a, b]$ and each of $\int_a^b G$, $\int_a^b H$ and $\int_a^b K$ exists, then the following statements are equivalent:*

(1) *the integral equation*

$$f(x) = w(x) + (LRLR)\int_a^x (fH + Gf + fKf)$$

has a bounded solution on $[a, b]$, and

(2) *the inverse of $v(x)$ exists and is bounded on $[a, b]$, where $u(x)$ and $v(x)$ are defined by the equation*

$$[u(x), v(x)] = [w(a), 1]_a\prod^x\left(\mathbf{I} + \begin{bmatrix} H & -K \\ dw & -G \end{bmatrix}\right)$$

for $a \leqslant x \leqslant b$.

Furthermore, if (1) *or* (2) *is true, then the only bounded solution of the integral equation in* (1) *is given by* $f(x) = [v(x)]^{-1}u(x)$ *for* $a \leqslant x \leqslant b$.

PROOF. To simplify notation, let $\mathbf{Q}$ denote the function such that

$$\mathbf{Q}(x,y) = \mathbf{I} + \begin{bmatrix} H(x,y) & -K(x,y) \\ w(y) - w(x) & -G(x,y) \end{bmatrix}$$

for $a \leqslant x < y \leqslant b$. Then, if we let

$$\mathbf{F}(x,y) = \begin{bmatrix} 0 & 0 \\ w(y) - w(x) & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{G}(x,y) = \begin{bmatrix} H(x,y) & -K(x,y) \\ 0 & -G(x,y) \end{bmatrix}$$

for $a \leqslant x < y \leqslant b$, it follows from Lemmas 5 and 8 that $\mathbf{Q}$ is in $OP°$ and OM^* on $[a, b]$.

PROOF [(1)$\rightarrow$(2)]. Suppose $a \leqslant x \leqslant b$. If $a = x$, then the result follows immediately. Therefore, suppose $a < x$. The existence of $(L)\int_a^x - fK$ follows from Lemma 2. Thus, $(L)\int_a^x(-G - fK)$ exists. Therefore, the existence of $(L)_a\prod^x(1 - G - fK)$ follows from Lemma 3. We now establish that

$$\left\{(L)_a\prod^x(1 - G - fK)\right\}[f(x), 1] = [u(x), v(x)].$$

Let $\varepsilon > 0$. If L denotes the interval function

$$L(r,s) = (LRLR)\int_r^s[fH + Gf + fKf] - [f(r)H(r,s) + G(r,s)f(s) + f(r)K(r,s)f(s)],$$

D is a subdivision of $[a, x]$ and $\{x_i\}_{i=0}^n$ is a refinement of D such that $(1 - G_i - f_{i-1}K_i)^{-1}$ exists for $i = 1, 2, \ldots, n$, then by employing an iteration technique used by B. W. Helton [11, Theorem 3.3], we have that

$$\left\{(L)_a\prod^x(1 - G - fK) + d_1\right\}[f_n, 1] = [f_0, 1]\left\{{}_a\prod^x\mathbf{Q} + \mathbf{d}_2 + \mathbf{d}_3\right\},$$

where

$$d_1 = \prod_{i=1}^{n} (1 - G_i - f_{i-1}K_i) - (L)_a\prod^x (1 - G - fK),$$

$$\mathbf{d}_2 = \prod_{i=1}^{n} \mathbf{Q}_i - {}_a\prod^x \mathbf{Q}$$

and

$$\mathbf{d}_3 = \prod_{i=1}^{n}\left(\mathbf{I} + \begin{bmatrix} H_i & -K_i \\ \Delta w_i & -G_i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ L_i & 0 \end{bmatrix}\right) - \prod_{i=1}^{n} \mathbf{Q}_i.$$

Thus,

$$\left\|\left\{(L)_a\prod^x (1 - G - fK)\right\}[f_n, 1] - [f_0, 1]_a\prod^x \mathbf{Q}\right\|$$

$$= \left\| -d_1[f_n, 1] + [f_0, 1]\{\mathbf{d}_2 + \mathbf{d}_3\}\right\|$$

(a) $$\leqslant |-d_1|\, \|[f_n, 1]\|$$

(b) $$+ \|\mathbf{d}_2\|\, \|[f_0, 1]\|$$

(c) $$+ \|\mathbf{d}_3\|\, \|[f_0, 1]\|.$$

Since $(1 - G)^{-1}$ exists on $[a, b]$, f is bounded on $[a, b]$ and each of $K(p, p^+)$ and $K(p^-, p)$ exists and is zero on $[a, b]$, it follows that there exists a subdivision D_1 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then $(1 - G_i - f_{i-1}K_i)^{-1}$ exists for $i = 1, 2, \ldots, n$. It follows from the existence of $(L)_a\Pi^x(1 - G - fK)$ that there exists a subdivision D_2 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then the expression in (a) is less than $\varepsilon/3$. It follows from the existence of ${}_a\Pi^x\mathbf{Q}$ that there exists a subdivision D_3 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_3, then the expression in (b) is less than $\varepsilon/3$. It follows from Lemma 9 and Theorem 1 that there exists a subdivision D_4 of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D_4, then the expression in (c) is less than $\varepsilon/3$.

Let D denote the subdivision $\cup_{i=1}^4 D_i$ of $[a, x]$. Thus, if $\{x_i\}_{i=0}^n$ is a refinement of D, then

$$\left\|\left\{(L)_a\prod^x (1 - G - fK)\right\}[f_n, 1] - [f_0, 1]_a\prod^x \mathbf{Q}\right\|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Hence,

$$\left\{(L)_a\prod^x (1 - G - fK)\right\}[f(x), 1] = [w(a), 1]_a\prod^x \mathbf{Q} = [u(x), v(x)].$$

From the definition of matrix equality, we have that

$$\left\{(L)_a\prod^x(1 - G - fK)\right\}f(x) = u(x)$$

and

$$(L)_a\prod^x(1 - G - fK) = v(x).$$

From Lemma 10, we have that $\{(L)_a\Pi^x(1 - G - fK)\}^{-1}$ exists. Thus, $[v(x)]^{-1}$ exists, and hence, $f(x) = [v(x)]^{-1}u(x)$. Also, note that Lemma 10 can be used to establish that v^{-1} is bounded on $[a, b]$. Therefore, (1) implies (2). Also, we have shown that, if f is a bounded solution to the integral equation in (1), then $f(x) = [v(x)]^{-1}u(x)$ for $a \leqslant x \leqslant b$.

PROOF [(2) $\rightarrow$ (1)]. Let $f(x) = [v(x)]^{-1}u(x)$ for $a \leqslant x \leqslant b$. Since v^{-1} and u are bounded on $[a, b]$, the funcion f is bounded on $[a, b]$. Suppose $a \leqslant x \leqslant b$. If $a = x$, then (1) follows immediately. Therefore, suppose $a < x$. We now establish that the equality in (1) is satisfied. Let $\varepsilon > 0$.

If D is a subdivision of $[a, x]$ and $\{x_i\}_{i=0}^n$ is a refinement of D, then by employing an iteration technique used by B. W. Helton [11, Theorem 3.2], we have that

$$f_n = w_n + \sum_{i=1}^n f_{i-1}H_i + \sum_{i=1}^n G_if_i + \sum_{i=1}^n f_{i-1}K_if_i + \sum_{i=1}^n \left(v_{i-1}^{-1}a_i - v_{i-1}^{-1}b_if_i\right),$$

where

$$[a_i, b_i] = [u_{i-1}, v_{i-1}]\left\{{}_{x_{i-1}}\prod^{x_i}\mathbf{Q} - \mathbf{Q}_i\right\}$$

for $i = 1, 2, \ldots, n$. To simplify notation, let $\mathbf{L}$, S, T, U and V denote the functions such that, if $a \leqslant r < s \leqslant x$, then

$$\mathbf{L}(r, s) = \begin{bmatrix} S(r, s) & T(r, s) \\ U(r, s) & V(r, s) \end{bmatrix} = {}_r\prod^s\mathbf{Q} - \mathbf{Q}(r, s).$$

Now, from the definitions of the functions involved, we have that

$$v_{i-1}^{-1}a_i = v_{i-1}^{-1}u_{i-1}S_i + U_i$$

and

$$-v_{i-1}^{-1}b_if_i = -v_{i-1}^{-1}u_{i-1}T_iv_i^{-1}u_i - V_iv_i^{-1}u_i.$$

Thus,

$$\left|f(x) - \left\{w(x) + \sum_{i=1}^{n} (f_{i-1}H_i + G_if_i + f_{i-1}K_if_i)\right\}\right|$$

$$= \left|\sum_{i=1}^{n} (v_{i-1}^{-1}a_i - v_{i-1}^{-1}b_if_i)\right|$$

(a) $$< \left|\sum_{i=1}^{n} v_{i-1}^{-1}u_{i-1}S_i\right|$$

(b) $$+ \left|\sum_{i=1}^{n} U_i\right|$$

(c) $$+ \left|\sum_{i=1}^{n} - v_{i-1}^{-1}u_{i-1}T_iv_i^{-1}u_i\right|$$

(d) $$+ \left|\sum_{i=1}^{n} - V_iv_i^{-1}u_i\right|.$$

It follows from Lemma 11 that **L** has bounded variation on $[a, x]$ and that $\int_r^s \mathbf{L} = \mathbf{0}$ for $a \leqslant r < s \leqslant x$. Thus,

$$0 = \int_r^s S = \int_r^s T = \int_r^s U = \int_r^s V$$

for $a \leqslant r < s \leqslant x$. Also, each of S, T, U and V has bounded variation on $[a, x]$. Thus, it follows from Lemma 1 that there exists a subdivision D of $[a, x]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then each of the expressions in (a), (b), (c) and (d) is less than $\varepsilon/4$. Hence, if $\{x_i\}_{i=0}^n$ is a refinement of D, then

$$\left|f(x) - \left\{w(x) + \sum_{i=1}^{n} (f_{i-1}H_i + G_if_i + f_{i-1}K_if_i)\right\}\right|$$

$$< \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon.$$

Hence, $(LRLR)\int_a^x (fH + Gf + fKf)$ exists and is $f(x) - w(x)$. Therefore, (2) implies (1). This completes the proof of Theorem 4.

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